The vacuum state of quantum gravity contains large-volume holes

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Abstract - We investigate, both analytically and numerically, a set of static metrics with the property that the integral of $\sqrt{g}R$ is zero. In a quantum theory of gravity based upon the Einstein action, these configurations will be present in the ground state as vacuum fluctuations (together with the spacetime foam). They contain a core with positive scalar curvature, surrounded by a shell with $R<0$. When the radius of the core approaches the Schwarzschild radius from below, the external radius increases, too, and the difference between the volume of the configuration and that of flat space diverges. The addition of a cosmological term to the action changes the total mass of these configurations. The combined effect of mass change and of the large internal volume considerably amplifies (locally) the cosmological term.

04.20.-q Classical general relativity.
04.60.-m Quantum gravity.

Although a complete and consistent theory of quantum gravity has not yet been established [1], it is possible to investigate some features of its ground state. We shall use in this investigation only standard, widely accepted elements, assuming that the dynamics of the gravitational field is defined by the Einstein action $S[g] = -(1/8\pi G)\int\sqrt{g}R d^4x$, and that the probability amplitude for any field configuration $g_{\mu\nu}(x)$ is given by the Feynman exponential $\exp\{iS[g]/\hbar\}$. (Below we shall use units in which $c=\hbar=1$, but in this introduction we leave $\hbar$ in the equations for clarity.)

The field configurations allowed in the classical theory minimize S and thus satisfy the Einstein equations. Throughout this paper we shall always consider the case of empty spacetime ($T_{\mu\nu}=0$); in this case, the solutions of Einstein equations are either flat spacetime $g_{\mu\nu}(x)=\eta_{\mu\nu}$, or gravitational waves, for which $R_{\mu\nu}(x)$ and $R(x)$ vanish at any point, and so does the lagrangian density $L=-(1/8\pi G)\sqrt{g}R$.

In the quantum theory, the mean value of any physical quantity is obtained in principle through an average over all configurations, with weighing factor $e^{iS/\hbar}$. This formal prescription is notoriously difficult to implement in a meaningful and consistent way. Implementation proposals include for instance the Euclidean lattice Regge calculus [2] and the "dynamical triangulations" [3]. The problems related with these techniques do not affect our present general analysis, however.

The quantum amplitude $e^{iS/\hbar}$ oscillates rapidly if $|S|>>\hbar$, and therefore off-shell configurations contribute very little to average values. A famous exception are the "spacetime foam" vacuum fluctuations [4], which are strong but confined to Planck-scale lengths $\approx l_P$; their curvature is such that $|S|\approx l_P^4|L(g)|\approx l_P^2\sqrt{g}|R|<\hbar$.

There is, however, another way to satisfy the condition $|S|<\hbar$. Consider more extensive field configurations which give opposite contributions to the action integral in different regions of spacetime. These configurations have vanishing or very small total action, even though their local curvature R may be such that $(1/8\pi G)\int\sqrt{g}|R|d^4x>\hbar$ over any single region. A cancellation between opposite contributions in different regions of spacetime can only occur because the gravitational action, unlike most other fundamental field actions, is not positive-defined.

Such extensive vacuum fluctuations are not confined to short distances. They have first been introduced in Ref. [5] in the framework of perturbation theory, and then re-considered in a more general context [6]. In a first-order weak field approximation the factor $\sqrt{g}$ in the action can be replaced by unity; the null-action condition then reduces to the requirement that the total integral of the scalar curvature R be zero, hence the name "dipolar" fluctuations.

In this work we find explicitly several new field configurations belonging to this set of "zero-modes" of the Einstein action. We go beyond the perturbative approximation and solve the full strong-field equations, finding almost-singular configurations with large local curvature and unbounded volume ($\sqrt{g}$>>1).

The problem of finding field configurations with zero total action is in principle unrelated to the Einstein equations, which are equivalent to minimize the lagrangian *density*. There is a mathematical trick, however, which helps imposing our integral condition while taking advantage of known solutions of the Einstein equations. We first write the Einstein equations with a source $T_{\mu\nu}$, namely

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) = -8\pi G T_{\mu\nu}(x) \qquad (i)$$

and their covariant trace

$$R(x) = 8\pi G g^{\mu\nu}(x) T_{\mu\nu}(x) \qquad (ii)$$

Then we consider a solution of equation (*i*) with any source $T_{\mu\nu}$ obeying the following integral condition

$$\int d^4x \sqrt{g(x)}\, g^{\mu\nu}(x) T_{\mu\nu}(x) = 0 \qquad (iii)$$

Taking into account (*ii*), we see that the pure Einstein action $S=-(1/8\pi G)\int\sqrt{g}R d^4x$ computed for this solution is zero. The source can be un-physical, because it does not represent any physical object, but is only a mathematical artifice allowing to find field configurations having certain pre-defined properties.

It is convenient to choose static and spatially symmetric configurations. Imagine a 3-sphere with constant or almost constant scalar curvature, surrounded by a shell with negative curvature. By adjusting the sizes of the two regions, and taking into account the volume factor $\sqrt{g}$, it is possible to obtain a total null action. Technically, this amounts to solve the Einstein equations in the static, spatially symmetric case with a virtual, unphysical source made of a central core with positive mass and an outer shell with negative mass. As we shall see, the situation is actually more complicated, as the density can vary in many ways and there is a vast choice of possible configurations even in this special symmetric case.

Our recourse to un-physical sources could be reminiscent of works on solutions of the Einstein equations with "exotic" matter sources (see for instance [7]) and of definitions of physically acceptable sources [8]. It should be clear from the discussion above, however, that our approach is different, and independent from specific assumptions on the sources. It is also essentially selfcontained, except for the two basic assumptions mentioned above (validity of the Einstein action and of the Feynman integral for gravitation).

1. Spherically symmetric metric tensor

Let us now find explicitly the metric of a dipolar fluctuation with spherical symmetry. As we shall see, there are several kinds of such fluctuations, having internal 4D volume either bounded or unbounded. We shall be especially interested into the unbounded-volume configurations, because they are much favoured in the presence of a cosmological/vacuum energy term.

Our starting point is the well-known expression of the Schwarzschild metric outside and inside a mass distribution with density $\rho(r)$ in the local inertial frame, namely

$$ds^2 = e^{2\phi(r)}dt^2 - \frac{dr^2}{1-2m(r)G/r} - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1.1}$$

This expression is quoted from Ref. [9], p. 608. We have changed the signs in front of dt^2, dr^2, $d\theta^2$ and $d\varphi^2$, in order to adapt the metric to our signature convention (+,-,-,-). We also have inserted explicitly the gravitational constant G, which is set to 1 in Ref. [9] through a suitable choice of the length unit. The function $m(r)$ represents the mass contained in a 3-sphere of radius r:

$$m(r) = \int_0^r 4\pi r'^2 \rho(r')dr' \tag{1.2}$$

The function $\phi(r)$ appearing in the g_{00} component of the metric ($g_{00}=e^{2\phi}$) is the solution of the differential equation

$$\frac{d\phi(r)}{dr} = \frac{[m(r)+4\pi r^3 p]G}{r[r-2m(r)G]} \tag{1.3}$$

where p is the pressure, which in the following we shall take to be zero. At the external boundary r_{ext} of the mass distribution, ϕ satisfies the condition $\phi(r_{ext})= ½ \ln(1-2MG/r_{ext})$, where M is the total mass, $M=m(r_{ext})$. We furthermore have $\phi(0)=0$.

All the relations above come from the solution of the Einstein equations with the given source, and are often employed in relativistic astrophysical stellar models. In that context, it is also necessary to specify relations between the source parameters ρ and p, compatible with the state equations of the stellar matter. In our case, ρ and p can be instead assigned arbitrarily, because the source can be unphysical. We shall assign ρ and p in such a way to obtain metrics for which the integral of the scalar curvature $\int\sqrt{g}Rd^4x$ is zero (zero-modes of Einstein action). This can be obtained in several different ways, ie there exist several different classes of zero-modes. We shall always consider the case of null pressure.

2. Zero-modes with virtual density ρ constant in absolute value and bounded volume

Let us consider a constant positive mass density ρ_0, and let r_0 be the corresponding Schwarzschild radius

$$r_0 = \sqrt{\frac{1}{\frac{8}{3}\pi\rho_0 G}} \tag{2.1}$$

The exact meaning of this quantity will soon be clear. It is the radius at which a singularity would arise, if the mass distribution were not cut before that point. Let us take r_1 and r_2 such that $r_1<r_0<r_2$ and define a density function $\rho(r)$ which is equal to ρ_0 up to r_1, becomes negative at r_1 and then zero from r_2 on:

$$\rho(r) = \rho_0 \varepsilon(r_1 - r)\theta(r_2 - r)\text{, with } \rho_0>0 \tag{2.2}$$

where ε and θ are the usual step functions. We define the adimensional quantities $a = \frac{r_1}{r_0}; b = \frac{r_2}{r_0}; a < 1, b > 1$.

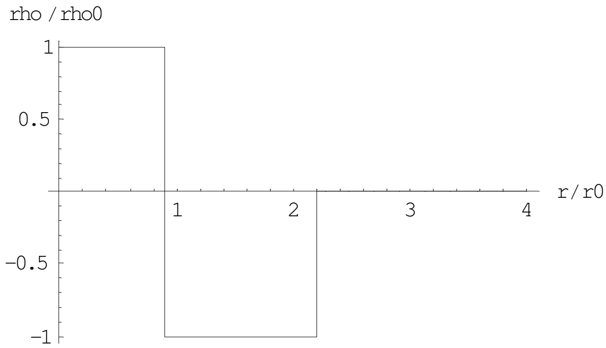


Fig. 1 - An example of mass density function with constant absolute value and adimensional parameters a=0.9, b=2.2.

For $r<r_1$ the mass function is

$$m(r) = \frac{4}{3}\pi\rho_0 r^3 \quad (r<r_1) \tag{2.3}$$

The integral of (1.3) gives

$$\phi(r) = -\frac{1}{4}\ln\left(1-\frac{r^2}{r_0^2}\right) \quad (r<r_1) \tag{2.4}$$

with r_0 as in (2.1). This allows to define the metric component $g_{00}(r)$ completely:

$$g_{00}(r) = e^{2\phi(r)} = \frac{1}{\left(1-\frac{r^2}{r_0^2}\right)^{1/2}} \quad (r<r_1) \tag{2.5}$$

Note the following useful relation between $g_{00}(r)$ and $g_{rr}(r)$:

$$g_{rr}{}^{-1}(r) = 1 - \frac{2m(r)G}{r} = 1 - \frac{8}{3}\pi\rho_0 r^2 = 1 - \frac{r^2}{r_0{}^2} = g_{00}{}^{-2}(r) \qquad (r<r_1) \tag{2.6}$$

The 4D invariant volume element is

$$\sqrt{|\det g(r)|} = \sqrt{g_{00}(r) g_{rr}(r) r^4 \sin^2\theta} = \frac{r^2 \sin\theta}{\left(1 - \frac{r^2}{r_0{}^2}\right)^{3/4}} \qquad (r<r_1) \tag{2.7}$$

and it is seen that its integral up to r_1 converges, when r_1 approaches r_0 from below. We saw in eq. (2.5) the meaning of r_0, as the point where the metric would become singular if the virtual mass density ρ were constant. But before arriving to r_0, we change the sign of ρ; we shall then define r_2, which until now has not been fixed, in such a way to satisfy the zero-mode condition $\int\sqrt{g}Rd^4x=0$. Note that r_1 is arbitrary, provided $r_1<r_0$. The zero-mode condition can be satisfied in several ways, but in order to become familiar with the solutions with un-bounded volume, let us suppose that r_1 is very close to r_0. The scalar curvature R is obtained by remembering that for any solution of Einstein equations, independently from the features of the source, one has $R=8\pi GT_{\mu\nu}g^{\mu\nu}$ and therefore in our case

$$R(r) = 8\pi G T_{\mu\nu}(r) g^{\mu\nu}(r) = 8\pi G\rho(r) g^{00}(r) = 8\pi G\rho(r)\left[g_{00}(r)\right]^{-1} \tag{2.8}$$

$$R(r) = 8\pi G\rho_0\left(1 - \frac{r^2}{r_0{}^2}\right)^{1/2} \qquad (r<r_1) \tag{2.9}$$

Now we can write the lagrangian density $\sqrt{g}R$ in our coordinates:

$$\sqrt{|\det g|}R(r) = \frac{8\pi G\rho(r) r^2 \sin\theta}{\sqrt{|g_{00}(r) g_{rr}{}^{-1}(r)|}} \qquad \text{(general expression)} \tag{2.10}$$

$$\sqrt{|\det g|}R(r) = \frac{8\pi G\rho_0 r^2 \sin\theta}{\left(1 - \frac{r^2}{r_0{}^2}\right)^{1/4}} \qquad (r<r_1) \tag{2.11}$$

All this is valid up to r_1, where ϕ attains the value

$$\phi(r_1) = -\frac{1}{4}\ln\left(1 - \frac{r_1{}^2}{r_0{}^2}\right) = -\frac{1}{4}\ln\left(1 - a^2\right) \tag{2.12}$$

and the mass function is

$$m(r_1) = \frac{4}{3}\pi\rho_0 r_1{}^3 \tag{2.13}$$

For $r>r_1$ the mass function is given by

$$m(r) = \frac{4}{3}\pi\rho_0(2r_1{}^3 - r^3) \qquad (r>r_1) \tag{2.14}$$

Inserting this into (1.3) and remembering that $\frac{8}{3}\pi\rho_0 G = \frac{1}{r_0{}^2}$, we obtain

$$\frac{d\phi(r)}{dr}=\frac{\frac{1}{2r_0^2}(2r_1^3-r^3)}{r\left[r-\frac{1}{r_0^2}(2r_1^3-r^3)\right]}=\frac{2a^3-s^3}{2r_0 s(s^3+s-2a^3)} \quad (s=r/r_0;\ r>r_1) \tag{2.15}$$

where $s=r/r_0$. By integrating this expression from r_1 to r, with the initial condition (2.12), we obtain the metric for $r>r_1$:

$$\phi(r)=\int_a^{r/r_0}\frac{2a^3-s^3}{2s(s^3+s-2a^3)}ds-\frac{1}{4}\ln\left(1-a^2\right) \quad (r>r_1) \tag{2.16}$$

This integral can be computed in closed form, but the result is quite bulky and we do not report it here. From $\phi(r)$ one obtains $g_{00}(r)$ just taking the exponential. The $g_{rr}(r)$ component of the metric is found directly from the mass function (2.14):

$$g_{rr}^{-1}(r)=1-\frac{2m(r)G}{r}=1-\left(\frac{2a^3}{s}-s^2\right) \qquad (s=r/r_0;\ r>r_1) \tag{2.17}$$

From this, through (2.10), one arrives at the lagrangian density $\sqrt{g}R$. (See also below, eq. (3.18), for a general expression of $\sqrt{g}R$.) We have plotted below the lagrangian density $\sqrt{g}R$ and the metric component g_{00} setting $r_0=1$ (ie the horizontal axis represents the ratio r/r_0), a=0.95 and $8\pi\rho_0 G=1$.

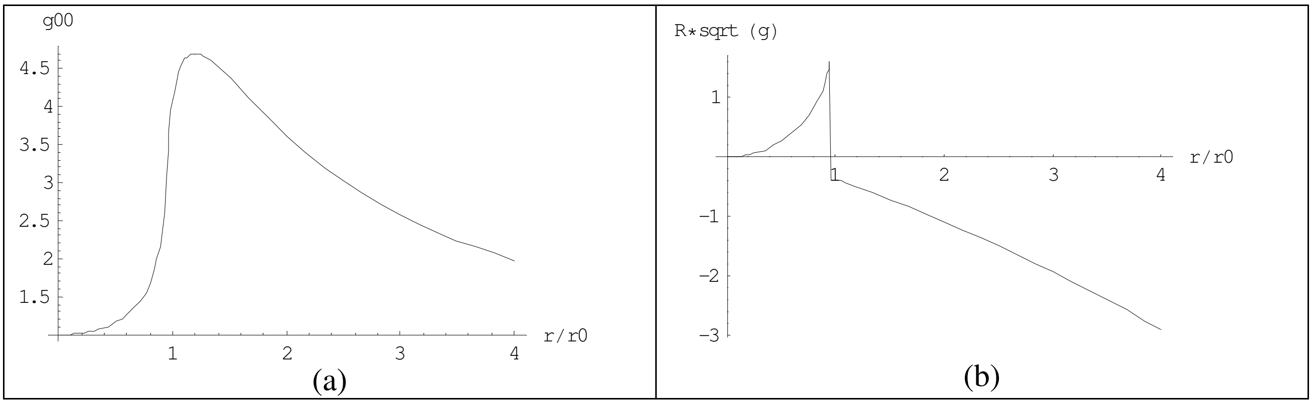


Fig. 2 – Graphs of the g_{00} metric element (a) and of the adimensional lagrangian density $\sqrt{g}R/(8\pi\rho_0 G)$ (b), as functions of the adimensional radius r/r_0, for the case of density ρ constant in absolute value, with "inversion radius" r_1 equal to the 95% of the Schwarzschild radius. (Compare the example in Fig. 1; here the external radius $b=r_2/r_0$ of the negative shell has not been represented, however, because it still has to be defined, according to the procedure below.)

So we are going to find, after fixing r_2, a zero-mode which looks like a field configuration having typical size r_0, generated by a positive spherical virtual mass surrounded by a negative spherical shell. The total mass of the source will not be zero (2.18). The source does not have any physical meaning, yet it is possible to visualize the formation of such a vacuum fluctuation by thinking of a condensation of positive mass up to a radius r_1, followed by deposition of a negative shell. Such fluctuations exist at any length scale. Given a length r_0, we can deduce the necessary virtual density ρ through eq. (2.1). For instance, at the atomic scale r_0~10^{-8} cm, remembering that G~10^{-66} cm^2 in natural units, we find that the necessary virtual density is ρ~10^{82} cm^{-4} ~ 10^{45} g/cm^3. This is clearly an enormous density and corresponds to very strong fields. However, such a high value for ρ is only needed when r_1 is close to r_0, so that a singularity is approached. At the same length scale, there are other less singular zero-modes which "change sign" earlier; their volume is much smaller, and so they will be less affected by a cosmological term, as we shall discuss in Section 4.

The procedure for the mathematical construction of a zero-mode next requires to set the integral of $\sqrt{g}R$ to zero and deduce from this that value of r_2, such that the negative part of the integral compensates the positive part. This is possible if the integral does not converge too fast, which has to be checked by examining the behaviour of the integrand function. In the example in Fig. 2 we see that r_2 is not far to the right of r_0. Substituting r_2 in the integral for $M(r)$, one obtains the total mass of the zero-mode:

$$M = m(r_2) = \int_0^{\infty} 4\pi r'^2 \rho(r')dr' = \int_0^{r_2} 4\pi r'^2 \rho_0 \varepsilon(r_1 - r')dr' \qquad (2.18)$$

Apart from possible physical interpretations, this mass has a precise technical meaning: the metric outside the source is a Schwarzschild metric with total mass M. We shall find that M is always negative. In our earlier perturbative calculations [5] we found a much simpler relation between M and the integral of the lagrangian density, namely $\int\sqrt{g}R = 4\pi MG\tau$, where τ is the lifetime of the fluctuation. This relation was a consequence of an approximation in which g_{00} and $\sqrt{g}$ are close to 1. It followed that the weak-field zero-modes are really "dipolar" fluctuations, ie they have M=0. For strong fields we find, instead, that the total mass is generally of the same magnitude order as the positive and negative components, and so can be very large for quasi-singular zero-modes. This is clearly seen by re-writing M as

$$M = \rho_0 r_0^3 \int_0^{r_2/r_0} 4\pi u^2 \varepsilon(r_1/r_0 - u)du \qquad (2.19)$$

where the integral in u is an adimensional quantity of order 1. We do not expect, however, to see any physical manifestations of these large negative virtual masses, homogeneously distributed in space and time. They are, at most, only one of the many contributions to the vacuum energy density, which notoriously though mysteriously manage to combine with each other giving a very small total cosmological constant [10].

3. Modes with variable density and unbounded volume

Now we make our treatment more general. Assuming that the virtual density $\rho(r)$ decreases, in absolute value, when r increases, one can obtain several different metrics. The simplest case is that of functions of the form $|\rho(r)| \propto 1/r^{\alpha}$, with $\alpha>0$. It is easy to check that for $\alpha \geq 2$ no singularity is formed. For α=1 we obtain a mode with unbounded volume. The same holds for $1<\alpha<2$. But let us repeat in detail the same steps of the previous section. The density of the virtual source is

$$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^{\alpha} \varepsilon(r_1 - r)\theta(r_2 - r) \qquad (3.1)$$

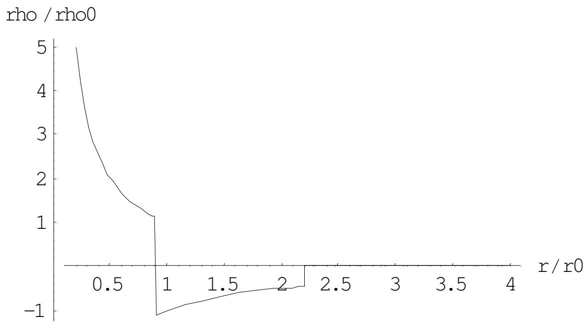


Fig. 3 – An example of variable density function with a=0.9, b=2.2, α=1.

The mass function for $r<r_1$ is

$$m(r) = 4\pi\rho_0 r_0^{\alpha} \frac{r^{3-\alpha}}{3-\alpha} \quad (r<r_1) \tag{3.2}$$

The inverse of the g_{rr} metric component is

$$g_{rr}^{-1}(r) = 1 - \frac{2m(r)G}{r} = 1 - \frac{8\pi G\rho_0}{3-\alpha} r_0^2 \left(\frac{r}{r_0}\right)^{2-\alpha} = 1 - \left(\frac{r}{r_0}\right)^{2-\alpha} \quad (r<r_1) \tag{3.3}$$

where in the last equality we have defined the Schwarzschild radius as before, ie

$$r_0 = \sqrt{\frac{3-\alpha}{8\pi\rho_0 G}} \tag{3.4}$$

It is convenient to write the differential equation for ϕ in the following form:

$$\frac{d\phi(r)}{dr} = \frac{m(r)G/r^2}{1-2m(r)G/r} \tag{3.5}$$

and note that

$$\frac{d}{dr}\frac{2m(r)G}{r} = 2G(2-\alpha)\frac{m(r)}{r^2} \quad (r<r_1) \tag{3.6}$$

It is straightforward to check that for $r<r_1$ the solution is

$$\phi(r) = -\frac{1}{2(2-\alpha)} \ln\left[1 - \left(\frac{r}{r_0}\right)^{2-\alpha}\right] \quad (r<r_1) \tag{3.7}$$

whence the metric is found:

$$g_{00}(r) = e^{2\phi(r)} = \frac{1}{\left[1 - \left(\frac{r}{r_0}\right)^{2-\alpha}\right]^{\frac{1}{2-\alpha}}} \quad (r<r_1) \tag{3.8}$$

The 4D volume element is

$$\sqrt{|\det g(r)|}=\sqrt{g_{00}(r)\,|\,g_{rr}(r)\,|}\,r^2\sin\theta = r^2\sin\theta\left[1-\left(\frac{r}{r_0}\right)^{2-\alpha}\right]^{\frac{\alpha-3}{2(2-\alpha)}} \qquad (r<r_1) \tag{3.9}$$

We would like to check if this is integrable when r approaches r_0 from below. Define $s=r/r_0$ as above and expand the term $\left(1-s^{2-\alpha}\right)$ as $\left(1+s^{1-\alpha/2}\right)\left(1-s^{1-\alpha/2}\right)$. Consider the integral

$$\int_0^1 \frac{1}{\left(1-s^{1-\alpha/2}\right)^{\frac{3-\alpha}{2(2-\alpha)}}}ds \tag{3.10}$$

The Taylor expansion of the denominator at s=1 gives to leading order $\left(1-s^{1-\alpha/2}\right)\approx(1-\alpha/2)(1-s)$ and the convergence rule gives

$$\frac{3-\alpha}{2(2-\alpha)}<1 \Rightarrow \alpha<1 \tag{3.11}$$

Therefore if the density decreases as $1/r$ (and not faster than $1/r^2$), the volume is unbounded when the inversion point r_1 of the sign of ρ approaches the Schwarzschild radius r_0 from below.

Finally let us write the lagrangian density $\sqrt{g}R$:

$$\sqrt{|\det g|}R(r)=\frac{8\pi G\rho(r)r^2\sin\theta}{\sqrt{g_{00}(r)\,|\,g_{rr}^{-1}(r)\,|}}=8\pi G\rho(r)r^2\sin\theta\left[1-\left(\frac{r}{r_0}\right)^{2-\alpha}\right]^{\frac{\alpha-1}{2(2-\alpha)}} \qquad (r<r_1) \tag{3.12}$$

In the interval 0≤α<2 this is integrable.

Next we solve the equation on the right of r_1. The boundary condition is

$$\phi(r_1)=-\frac{1}{2(2-\alpha)}\ln\left[1-\left(\frac{r_1}{r_0}\right)^{2-\alpha}\right]=-\frac{1}{2(2-\alpha)}\ln\left(1-a^{2-\alpha}\right) \tag{3.13}$$

For $r>r_1$ the mass function is

$$m(r)=\frac{4\pi\rho_0 G r_0^{\alpha}}{3-\alpha}(2r_1^{3-\alpha}-r^{3-\alpha}) \quad (r>r_1) \tag{3.14}$$

Remembering the definition of r_0 we obtain

$$\frac{2Gm(r)}{r}=2\frac{a^{3-\alpha}}{s}-s^{2-\alpha} \qquad (r>r_1) \tag{3.15}$$

The equation for ϕ is written

$$\frac{d\phi(r)}{dr}=\frac{m(r)G/r^2}{1-2m(r)G/r}=\frac{\frac{1}{2}\left(2a^{3-\alpha}-s^{3-\alpha}\right)}{s^2-2a^{3-\alpha}s+s^{4-\alpha}} \quad (r>r_1) \tag{3.16}$$

where $r=sr_0$. Integrating this expression from r_1 to r, with the initial condition (3.13), we obtain the metric to the right of r_1:

$$\phi(r)=\int_a^{r/r_0}\frac{2a^{3-\alpha}-s^{3-\alpha}}{2(s^2+s^{4-\alpha}-2a^{3-\alpha}s)}ds-\frac{1}{2(2-\alpha)}\ln\left(1-a^{2-\alpha}\right)\quad (r>r_1) \tag{3.17}$$

$$\sqrt{|\det g|}R(r)=\frac{8\pi G\rho(r)r^2\sin\theta}{\sqrt{g_{00}(r)\,|\,g_{rr}^{-1}(r)\,|}}=\frac{8\pi G(-\rho_0)r_0^{\alpha}r^{2-\alpha}\sin\theta}{\sqrt{g_{00}(r)\left|1-\frac{2a^{3-\alpha}r_0}{r}+\left(\frac{r}{r_0}\right)^{2-\alpha}\right|}}\quad (r>r_1) \tag{3.18}$$

Some graphs of the functions g_{00} and $\sqrt{g}R$ are given below, for six values of α. These are obtained by numerical integration of the differential equations. (For integer values of α the integral (3.17) for $\phi(r)$ can also be computed explicitly, but the result is quite bulky.) Like for the solutions with constant virtual density (Fig. 2), the quantity on the horizontal axis is the reduced radius r/r_0 and the sign of the virtual density is inverted at a distance r_1 such that $a=r_1/r_0=0.95$ (Fig.s 4, 5) or $a=0.99$ (Fig.s 6, 7).

Note that in Fig. 4 the mass distribution of the outer shell has not yet been cut-off at the radius r_2 such that the integral of $\sqrt{g}R$ is zero. In Fig. 5, r_2 would fall outside the depicted region; it is possible to show, by enlarging the scale, that $|\sqrt{g}R|$ continues to grow almost linearly. It is also apparent from the graphs that r_2 shifts more and more to the right when α increases. This means that the negative outer shell is much larger than the positive core when α approaches 2 and the virtual density decreases quickly with r. Numerical estimates are given in Sect. 5.

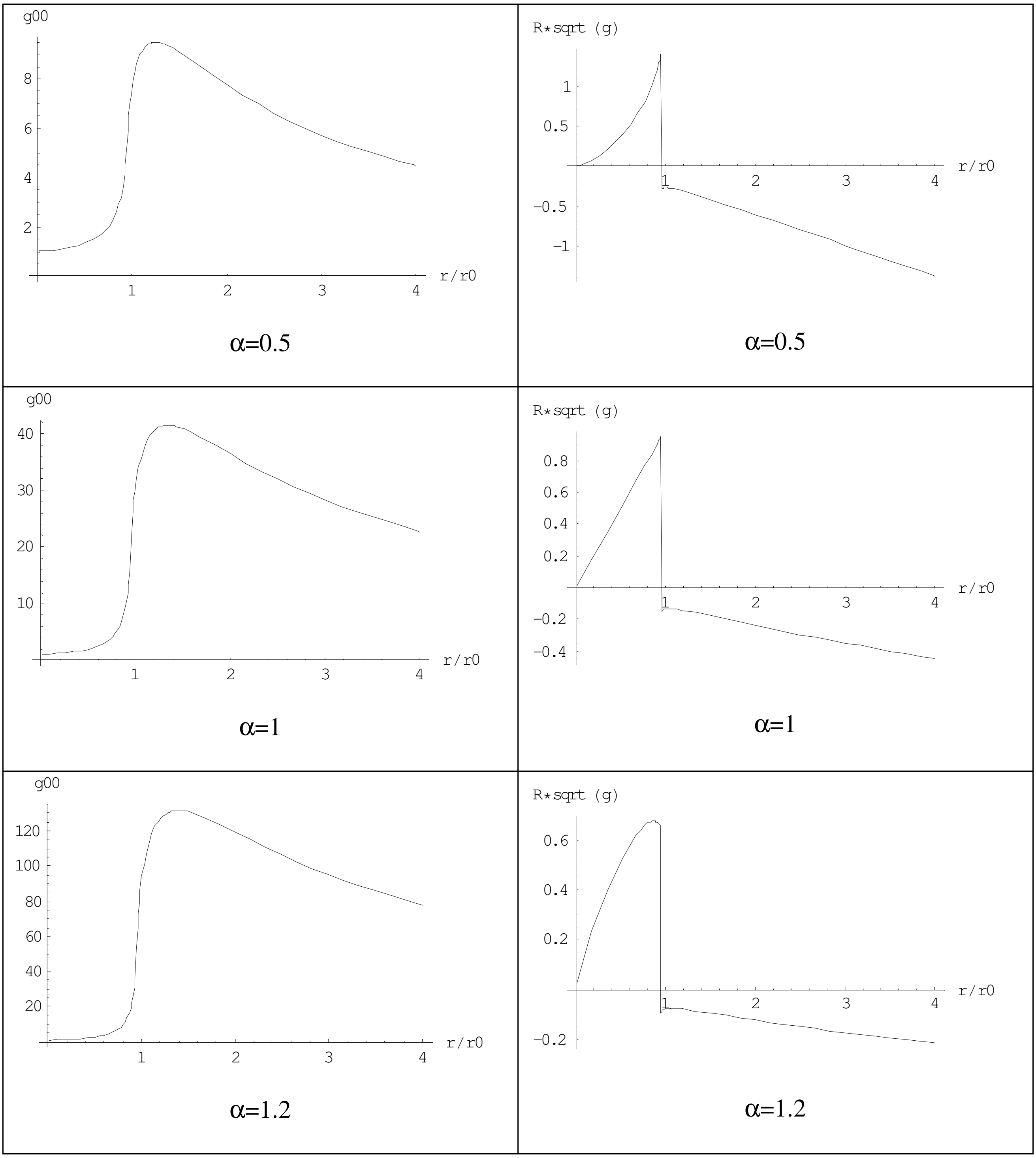


Fig. 4 – Component 00 of the metric and lagrangian density with *a*=0.95.

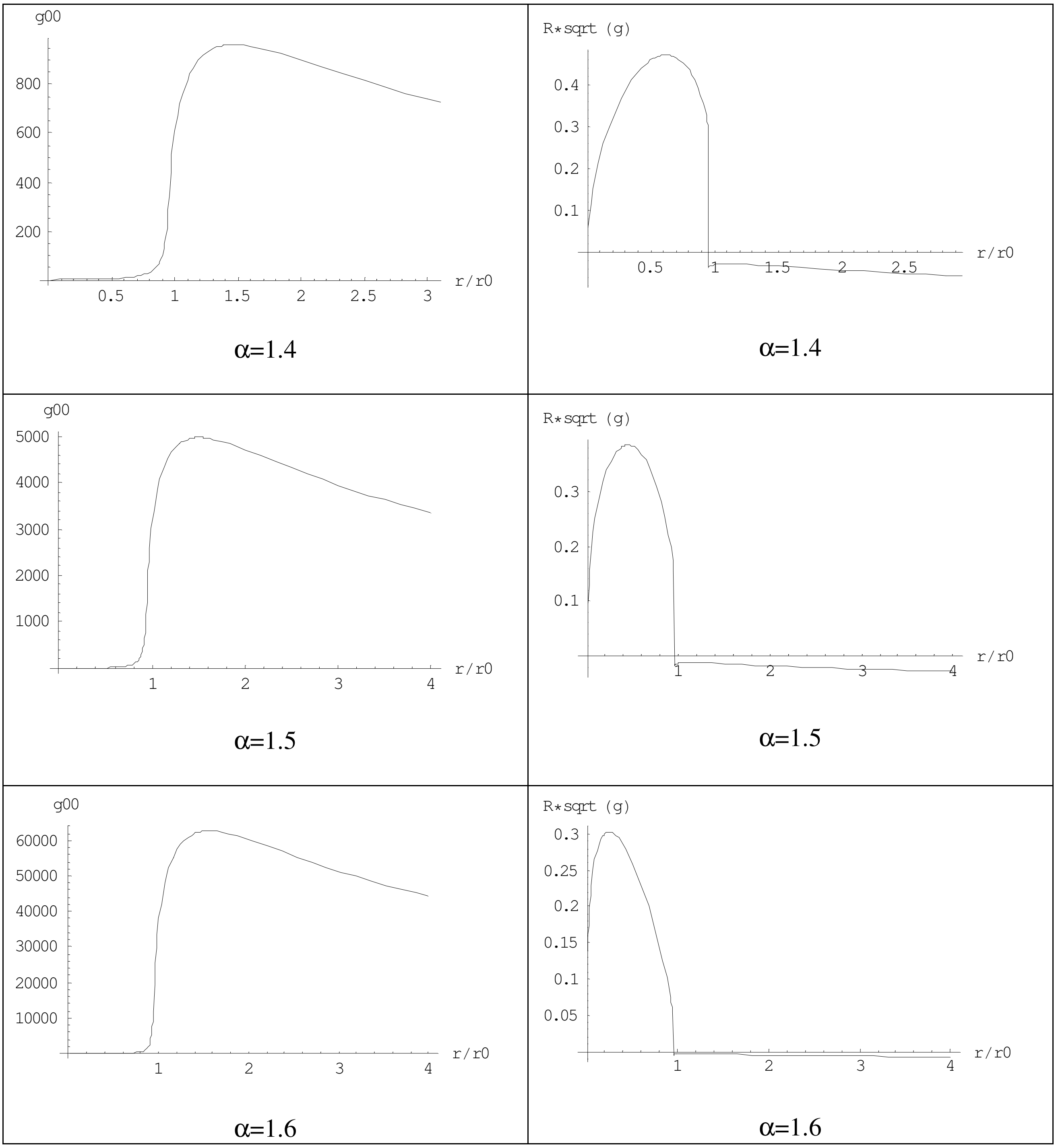


Fig. 5 – Component 00 of the metric and lagrangian density with *a*=0.95.

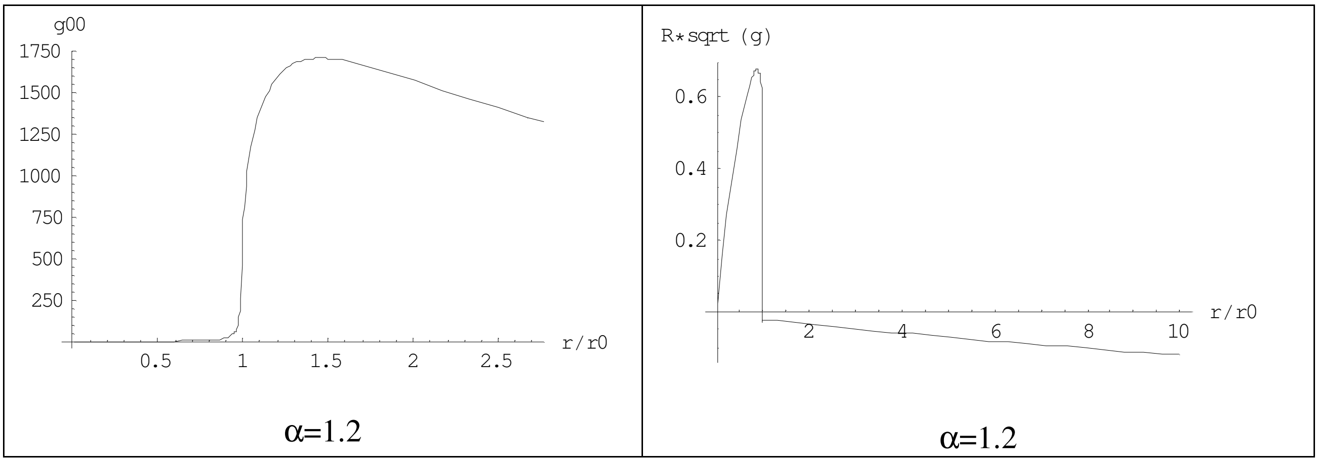


Fig. 6 – Component 00 of the metric and lagrangian density with *a*=0.99.

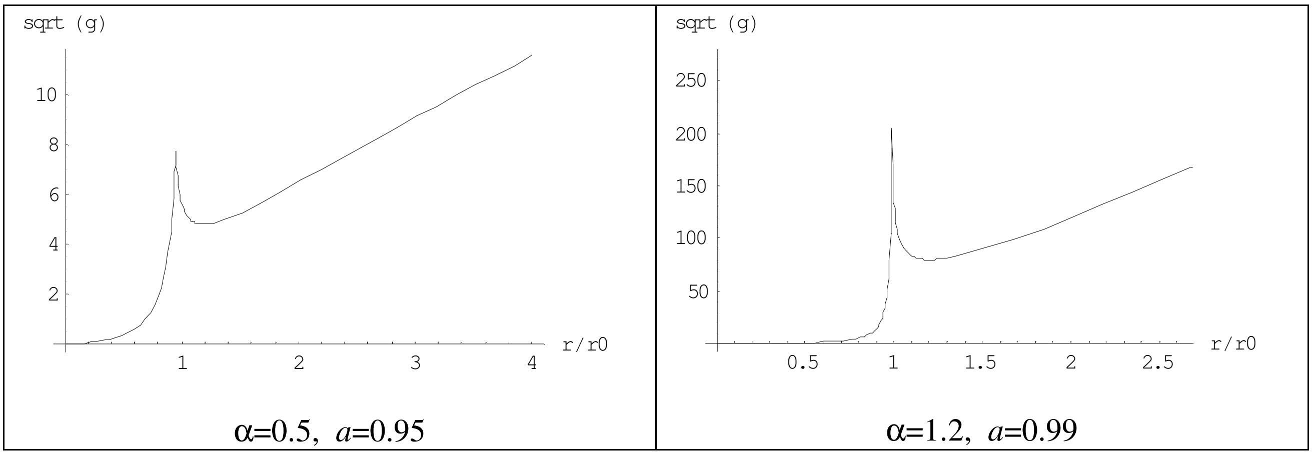


Fig. 7 – 4D volume element $\sqrt{g}$ showing the volume increase near the singularity. In the first case the divergence is integrable when $a \rightarrow 1$, in the second case it is not integrable.

Let us re-write the zero-mode condition in a way suitable for generalization to the case when a cosmological constant is present. In the integral of $\sqrt{g}R$ factorize a term τG, where τ is the time duration of the fluctuation, and write formally as follows

$$\int \sqrt{g(x)} R(x) d^4x = \tau G \int_0^{r_2} f(r) dr \tag{3.19}$$

Denoting by F the primitive of f, the null-action condition is

$$\tau G \big(F(r_2) - F(0) \big) = 0 \tag{3.20}$$

which allows to determine r_2. The parameter r_2, in turn, allows to compute the total mass M (eq. (3.14)), with an expression that we can formally write as follows:

$$M = m(r_2) = \int_0^{r_2} h(r) dr = H(r_2) - H(0) \tag{3.21}$$

where H is the primitive of h.

4. Change in the zero-mode condition when a cosmological term is present

Let us now introduce into the gravitational action a cosmological term, or vacuum energy term. The cosmological term and the R^2 terms (see below, Sect. 6) are the most common invariant additions to the pure Einstein action, so it is natural to consider them here. In our previous work on dipolar fluctuations in the perturbative approach, we pointed out that a cosmological term affects the dipolar fluctuations. We noticed that the influence of the cosmological term upon dipolar fluctuations could actually be more relevant than its usual "classical" effects (large-scale spacetime curvature, etc.).

We have a further special motivation for introducing the cosmological term: we have shown that a *local* vacuum-like energy density is present in condensed matter systems described by a macroscopic order parameter [11]. We can expect that a local vacuum energy density term breaks the symmetry of the dipolar fluctuations of the pure Einstein action. As we stressed above, these fluctuations can be very large, but are homogeneously distributed in spacetime and therefore unobservable.

Our present non-perturbative approach gives us a more general insight into the effect of the vacuum energy density term. In the gravitational action with a Λ-term, namely

$$S[g_{\mu\nu}(x)] = -\frac{1}{8\pi G}\int \sqrt{g(x)}R(x)d^4x + \frac{\Lambda}{8\pi G}\int \sqrt{g(x)}d^4x \tag{4.1}$$

the factor Λ multiplies the 4D volume of spacetime. In perturbation theory, the background spacetime is taken to be flat. Here, we are instead considering strongly curved and almost singular geometries, like "holes" in spacetime having an internal volume much larger than the volume they occupy as seen from the outside. These field configurations can only exist as vacuum fluctuations, because they have null action but their sources are un-physical.

Let us define better the distinction between "internal" and "external" volume. The internal volume is the real volume given by the integral of $\sqrt{g}$ from $r=0$ to the external boundary of the source at $r=r_2$. If we look at the whole field configuration, for $r>r_2$ we must describe it as a spherically symmetric metric generated by a source of radius r_2 and total mass M given by eq.s (3.14), (3.21). We could call it a Schwarzschild metric, were it not for the fact that M is negative (because tipically $r_2>>r_1$).

For $r>r_2$, the scalar curvature is zero and it is clear from the general expression of the spherically symmetric metric in vacuum that the volume element $\sqrt{g}$ is the same as in flat space, because $g_{00}g_{rr}=1$. Judging from its field, what we see is therefore a source with mass M which occupies a space equal to $V_{ext}=(4\pi/3)r_2^3$. We define the source density in a natural way as the ratio M/V_{ext}. As remarked in Sect. 2, this density can be very large, but it can not have any physical consequences, since these sources are just vacuum fluctuations which "pop up" everywhere in spacetime with the same probability.

Consider how these fluctuations and their mass are modified by the presence of a local cosmological term Λ. The condition $S[g_{\mu\nu}]=0$ now implies a different value for the outer radius r_2, and therefore for the total mass. The variation ΔM is a function of the local Λ, and as such can be observed. This is a crucial point of our analysis, a point of possible contact between microscopic quantities and observable parameters.

In the presence of the Λ-term, the zero-mode condition (3.20) becomes

$$\tau G\big(F(r_2') - F(0)\big) = \Lambda \Delta V^{(4)} \tag{4.2}$$

where r_2' denotes the new value of the outer radius r_2 which satisfies the condition and $\Delta V^{(4)}=\tau\Delta V^{(3)}$ is the 4D volume difference between the considered configuration $g_{\mu\nu}$ and flat space. (Thus the volume $V^{(3)}$ of the fluctuation would exactly be $\Delta V^{(3)}+V_{ext}$, but we shall see that $\Delta V^{(3)}$ is always much larger than V_{ext}.) After defining $\Delta r_2=r_2'-r_2$, by subtraction of the two conditions (4.2) and (3.20) we obtain

$$\frac{F(r_2') - F(r_2)}{\Delta r_2} = \frac{\Lambda \Delta V^{(3)}}{G \Delta r_2} \tag{4.3}$$

whence for small Δr_2, remembering that $F'(r_2)=f(r_2)$, one has

$$\Delta r_2 = \frac{\Lambda \Delta V^{(3)}}{G f(r_2)} \tag{4.4}$$

The total mass M' of the new configuration must be computed with the new outer radius r_2': $M'=m(r_2')=H(r_2')-H(0)$. We can write the mass variation as follows:

$$M' - M = \Delta M = h(r_2)\Delta r_2 = \frac{\Lambda}{G} \Delta V^{(3)} \frac{h(r_2)}{f(r_2)} \tag{4.5}$$

It is straightforward to see that

$$\frac{h(r)}{f(r)} = \frac{1}{8\pi}\sqrt{g_{00}(r) g_{rr}^{-1}(r)} = \frac{1}{8\pi}\sqrt{g_{00}(r)\left[1 - \frac{2a^{3-\alpha} r_0}{r} + \left(\frac{r}{r_0}\right)^{2-\alpha}\right]} \quad (r>r_1) \tag{4.6}$$

This function of r turns out to be, for our solutions, always such that $h(r_2)/f(r_2)>>1$. For instance, for α=1.7 and a=0.99 we obtain (compare Sect. 5, "Numerical evaluation") the behaviour shown in Fig. 8, at two different scales.

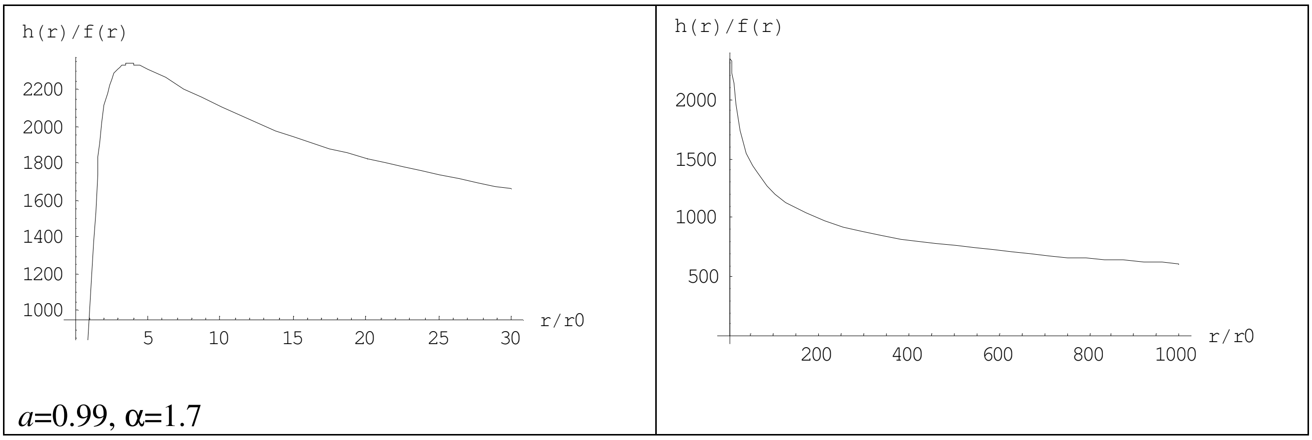


Fig. 8 - The function $h(r)/f(r)$ defined in eq. (4.6), for the case α=1.7 and a=0.99.

Dividing both members by V_{ext} in (4.5) we find

$$\frac{\Delta M}{V_{ext}} = \frac{\Lambda}{G} \frac{\Delta V^{(3)}}{V_{ext}} \frac{h(r_2)}{f(r_2)} \tag{4.7}$$

The term on the left represents the effective density variation of a fluctuation, "seen from the outside", when a Λ term is present. On the right, $\Delta V^{(3)}$ is the difference between the real internal volume of the configuration and $V_{ext}=(4\pi/3)r_2^3$. Remembering that $V^{(3)} >> V_{ext}$ for the fluctuations we are interested in, we can re-write the ratio approximately as $V^{(3)}/V_{ext}$, and it is very large for those configurations. The ratio Λ/G is the mass-energy density equivalent of the Λ-term, which is measured in natural units in cm^{-4} and in the SI in J/m^3.

We recall that the presently accepted value for Λ/G at the cosmological scale is of the order of 10^{16} cm^{-4}, or 0.1 J/m^3. A "local" vacuum-energy-like term Λ_{loc} due to coherent material fields [11] can be considerably larger, up to $\Lambda_{loc}/G \approx 10^8$ J/m^3. Such density is still too small to cause any appreciable gravitational effect at the classical level, but its local effect upon the density of dipolar vacuum fluctuations is amplified, according to eq. (4.7), by the factors $h(r_2)/f(r_2)$ and $V^{(3)}/V_{ext}$. We know from the Casimir effect that local variations in the vacuum fluctuations are observable; and these gravitational fluctuations appear to be much stronger than the fluctuations in quantum electrodynamics, because of the peculiar non-positive-definite nature of the gravitational action.

Unlike real mass density, variations in the virtual mass density can oscillate fast, following the time variations of Λ_{loc}. It is possible to show, for instance, that in superconductors described by a macroscopic wave function ψ, Λ_{loc} depends on $|\psi|$ and $|\nabla\psi|$. In certain conditions, $|\psi|$ and $|\nabla\psi|$ can easily oscillate in time with frequencies of the order of MHz-GHz. This variable Λ_{loc} amounts classically to a small oscillating mass-energy density, which generates in principle a small gravitational field, too small for detection. In the quantum theory, however, an oscillating Λ_{loc} induces much larger variations in the virtual mass density of dipolar vacuum fluctuations. If we compute transition matrix elements in quantum gravity (even just perturbatively) between the vacuum state and states with one or more virtual gravitons of energy-momentum E, P, the oscillations of $|\psi|$ and $|\nabla\psi|$ play the role of a periodic perturbation and the usual relations hold between the perturbation frequency and E. The factor $V^{(3)}/V_{ext}$ amplifies the perturbation and thus the transition amplitudes. This mechanism will be studied in more detail in a future work.

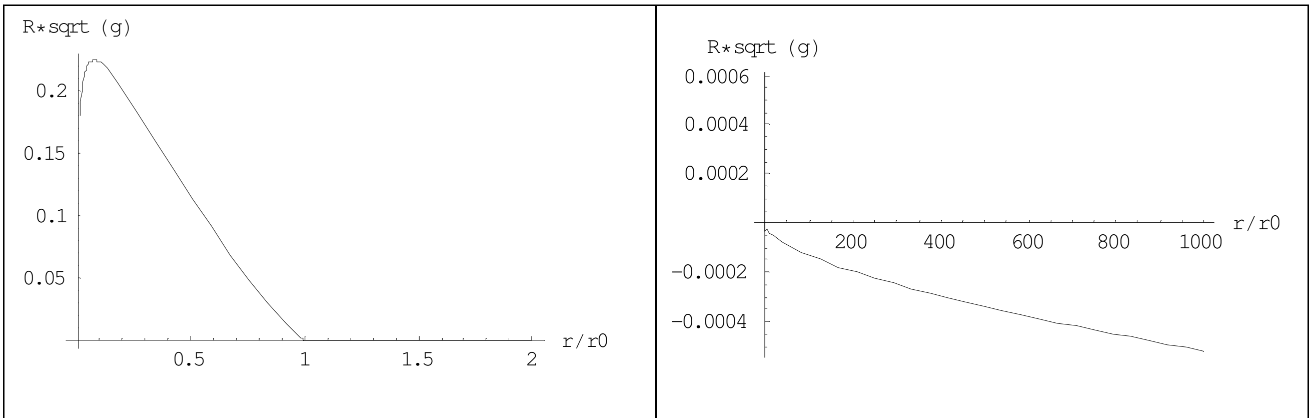
5. Numerical evaluation

In this section we give an example of numerical solutions of the differential equations for the zero-modes. Outside the inner radius r_1, analytical solutions are not available; there is a formal integral expression (eq. (3.17)), but the primitive can only be found explicitly for integer values of the exponent α. (α defines the behavior of the virtual density: $|\rho|/\rho_0=(r_0/r)^\alpha$.) It is therefore convenient to solve the differential equation numerically from the beginning with the Runge-Kutta method. This can be done quickly for several values of α, thus obtaining an insight of the behavior of the corresponding solutions. The radius r_1 is always smaller than the Schwarzschild radius r_0, so that singularities are avoided; in terms of the adimensional parameter $a=r_1/r_0$, this means $a<1$.

We saw that when a approaches 1 from below and $1<\alpha<2$, the volume of the core is unbounded and diverges like a power of $1/(1-a)$. Such large-volume fluctuations are most relevant when a cosmological term Λ is present in the action. The ensuing "amplification" of the Λ-term (eq. (4.7)) also depends on the ratio of the metric-related functions $h(r)$ and $f(r)$ evaluated at the cut-off point r_2. Therefore the crucial points to be checked through numerical evaluation are the following.

(1) Check that it is always possible to satisfy the zero-mode condition, ie the radius r_2 exists. The answer is positive in all cases. The lagrangian density $\sqrt{g}R$ grows almost linearly for large values of r, so it is quite simple to estimate r_2. One generally finds that for the configurations more close to the singularity (a close to 1) the radius r_2 is much larger than the core radius r_1. The numerical integration of $\sqrt{g}R$ (which is itself a numerical solution of a differential equation) can be done most safely by separating the two regions $r<r_1$ and $r>r_1$. The discontinuity at $r=r_1$ does not represent a problem. In order to determine r_2, we first find the integral from $r=0$ to $r=r_1$, then we make a polynomial fit of $\sqrt{g}R$ for $r>r_1$ and impose that the integral of the polynomial cancels the first integral. This yields an approximate value of r_2. Then we can perform the real numerical integration up to this value "$\pm\varepsilon$", and determine a safe error interval ε by checking that the integral changes sign between $r_2+\varepsilon$ and $r_2-\varepsilon$. Actually, for our present purposes it is sufficient to find a lower limit for r_2 – see Point 2 below.

For instance, with a=0.99 and α=1.7 the lagrangian density $\sqrt{g}R$ has the following behaviour (Fig. 9, first until r/r_0=2 and then at larger distances):



a=0.99, α=1.7

Fig. 9 – Behavior of the lagrangian density $\sqrt{g}R$ for a=0.99, α=1.7. At large distances, this is well approximated by a quadratic function.

The integral from 0 to 0.99 gives +0.115. A quadratic fit of the function at large distances, based on the points (0;0), (100; -0.00013), (300; -0.00025), (500; -0.00034), (800; -0.00045), gives

$$R\sqrt{g} \approx - 9\cdot 10^{-7}\,(r/r_0) + 5\cdot 10^{-10}\,(r/r_0)^2 \qquad (5.1)$$

Imposing that the integral of this function equals -0.115, we find that r_2/r_0 must be larger than 500.

(2) Evaluate the ratio between external and internal volume of the configuration. This ratio has been defined in Sect. 4 and equals the average of the ratio between the volume element $\sqrt{g}dr$ of the fluctuation and the volume element of flat space $4\pi r^2 dr$, in the interval (0, r_2). One always observes that this ratio has a peak at $r=r_1$, as anticipated (see Fig. 7) The height of the peak depends on the regulator (1-a). This "peak inside the core" does not make, however, the dominant contribution to the volume of the fluctuation. The dominant contribution comes from the region just to the right of r_1, where we do not know the analytic form of the metric. The reason for this, as can be seen from the graphs of the metric, is that g_{00} reaches its maximum to the right of r_1, and then decreases due to the effect of the negative density outside the core.

Going back to our example of Fig. 9, we find that the volume ratio $\sqrt{g}/r^2$ (we omitted the 4π in the calculations and in Table 1) in the interval (0,0.99) is equal to 4380; the integral from 0.99 to 500 is $2.55 \cdot 10^6$, therefore the average of $\sqrt{g}/r^2$ is ca. $5.1 \cdot 10^3$.

(3) Evaluate the function $h(r)/f(r)$ at $r=r_2$. (Compare Fig. 8.) In our example, we find that this is of the order of 500.

Note that the overall amplification factor $(\sqrt{g}/r^2)[h(r)/f(r)]$ will be under-estimated whenever r_2 is over-estimated, because both the average volume ratio $\sqrt{g}/r^2$ and the metric ratio $h(r_2)/f(r_2)$ decrease with r, when $r >> r_1$. Taking advantage of this, we used a graphical approximation to evaluate over-estimated values of r_2 and thus under-estimated values of the ratios $\sqrt{g}/r^2$ and h/f and of the overall amplification factor. These are given in Table 1. Any value can be checked through the more elaborate fit procedure mentioned above. See, for instance, how the values for a=0.99 and α=1.7 compare with the exact values.

a	α	r_2/r_0	$\sqrt{g}/r^2$	h/f	Amplification
0.99	0.5	4	3	0.5	1.5
	0.7	4	5	0.6	3
	1	5	8	1.2	10
	1.2	12	15	2	30
	1.3	18	20	3.5	70
	1.5	50	100	18	$1.8 \cdot 10^3$
	1.7	$2 \cdot 10^3$	$1.5 \cdot 10^3$	500	10^6
0.999	0.5	6	5	1.5	7.5
	1	12	20	5	100
	1.3	60	60	25	$1.5 \cdot 10^3$
	1.5	300	300	200	$6 \cdot 10^4$
	1.7	$3 \cdot 10^4$	$3 \cdot 10^4$	$2 \cdot 10^4$	$6 \cdot 10^8$
0.9999	1	30	30	25	750
	1.5	$1.3 \cdot 10^3$	$2 \cdot 10^3$	$2 \cdot 10^3$	$4 \cdot 10^6$
	1.7	$3 \cdot 10^5$	$5 \cdot 10^5$	$5 \cdot 10^5$	$2.5 \cdot 10^{11}$

Table 1 - Under-estimated values for the amplification factor of the cosmological term in a dipolar vacuum fluctuation (eq. (4.7)). a is the regulator, ie the ratio of the fluctuation core r_1 to the Schwarzschild radius r_0. α is the exponent determining the spatial decrease of the virtual mass density. r_2 is the radius of the outer negative shell, such that the total action of the fluctuation is zero. $\sqrt{g}/r^2$ is the average ratio of the internal volume element to the flat volume element. h/f is an additional amplification factor arising from the metric components evaluated at r_2. The total amplification factor can be very large for α close to 2, but this also implies that the outer radius r_2 is much larger than the core $r_1 \sim r_0$.

6. Final remarks

It is natural to ask whether, and to which extent, the presence of R^2 terms in the gravitational action changes the above picture. The R^2 terms are thought to arise in any effective action of quantum gravity, due to quantum corrections. They are positive-definite and can not cancel each other in different regions like the Einstein R term. Numerical simulations usually comprise the R^2 terms.

They are relevant only at very small scales, and we shall see that for dipolar fluctuations of atomic/nuclear size they can be disregarded.

In general, a term ηR^2 in the action changes the field configurations described above by increasing or decreasing their external radius r_2. If the coefficient η is positive, r_2 increases, because the integral of $\sqrt{g}R$ needs to "pick up" some more negative curvature from the outer shell in order to yield a null action. The total mass of the configuration is therefore slightly increased. This effect, however, is re-absorbed in the unobservable contribution to the homogeneous virtual mass density. For a numerical estimate, consider for instance a fluctuation with outer radius r_2 of the order of the Bohr radius, namely $r_2=10^{-8}$ cm. With regulator a=0.999 and exponent α=1.7, the amplification factor is at least $6\cdot10^8$ (Table 1). The ratio r_2/r_0 is at most $3\cdot10^4$, so the core of the fluctuation is larger than an atomic nucleus. The scalar curvature R is at most of order $r_1^{-2} \approx 10^{25}$ cm^{-2}. The relative magnitude of the Einstein term R/G and the R^2 terms ηR^2 is $(\eta RG)^{-1} \approx \eta^{-1}\cdot10^{41}$, where we put $G\approx10^{-66}$ cm^2 in natural units. η denotes the adimensional coefficient of the R^2 term, which is unknown, but has been estimated in the simulations by Hamber [2] at less than unity. It follows that the R^2 terms can be disregarded in comparison to the Einstein term at this length scale.

It is also interesting to note the magnitude order of the virtual mass density variation predicted, with the parameters above, in the presence of a local vacuum energy term with $\Lambda_{loc}/G \approx 10^8$ J/m^3 (Sect. 4). The conversion factor $1/c^2$ to mass density and the adimensional amplification factor together give a density variation $\Delta M/V_{ext} \approx 1$ kg/m^3. (And the amplification factor can easily be larger, for smaller cores or more singular configurations.) This variation follows the time behaviour of Λ_{loc}, possibly at high frequency and with sign inversion. The potential interest of these fluctuations as sources in processes involving virtual gravitons is therefore clear.

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